DISCRETE INTERVAL–VALUED CHOQUET INTEGRALS

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Summary

Based on different linear orders on intervals, several approaches to discrete interval–valued Choquet integrals are introduced and compared. We also present an application to multicriteria decision making problems.

Keywords: Choquet integral, expected value, interval–valued fuzzy set.

1 INTRODUCTION

For a fixed finite universe $U = \{u_1, \dots, u_n\}$, a fuzzy subset F of U is given by its membership function $F : U \to [0,1]$ (we will not distinguish fuzzy subsets and the corresponding membership functions notations). For several practical purposes, especially in multicriteria decision making, the expected value $E(F)$ of F should be introduced. The original Zadeh approach in [11] was based on a probability measure P on U, $P(u_i) = p_i$, and then $E(F) = \sum_{i=1}^{n} p_i\, F(u_i)$. More general approach, not limited by the non–interaction of single elements of U, is based on a fuzzy measure $m : 2^U \to [0,1]$, $m(\emptyset) = 0$, $m(U) = 1$, $m(A) \le m(B)$ whenever $A \subseteq B \subseteq U$ and the Choquet integral [3, 4],

$$\begin{aligned} E(F) = C_m(F) = \\ = \sum_{i=1}^{n} F(u_{\sigma(i)})(m(\{u_{\sigma(i)}, \dots, u_{\sigma(n)}\}) - \\ - m(\{u_{\sigma(i+1)}, \dots, u_{\sigma(n)}\})), \quad (1) \end{aligned}$$

where $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ is a permutation so that $F(u_{\sigma(1)}) \le F(u_{\sigma(2)}) \le \cdots \le F(u_{\sigma(n)})$, and $\{u_{\sigma(n+1)}, u_{\sigma(n)}\} = \emptyset$ by convention.

A further generalization of fuzzy sets into interval–valued fuzzy sets and Atanassov's intuitionistic fuzzy sets (these two concepts are isomorphic and thus we will discuss interval–valued fuzzy sets only) has brought the necessity to introduce the expected value also for these objects. Recall that an interval–valued fuzzy set F is characterized by its membership function $F : U \to J([0,1])$, where $J([0,1]) = \{[a,b]|, 0 \le a \le b \le 1\}$.

The aim of this paper is to discuss the expected value of interval–valued fuzzy sets based on the concept of discrete interval–valued Choquet integral.

The paper is organized as follows. In the next section, standard approach to the discrete interval–valued Choquet integral arising from the concept of Aumann integral [1] is recalled. In Section 3, several alternative approaches based on various linear orders on intervals are proposed. In Section 4, some examples of the proposed integrals and their relationships are introduced. Finally, some concluding remarks are added.

2 STANDARD DISCRETE INTERVAL–VALUED CHOQUET INTEGRAL

Generalization of reals into (closed real) intervals was forced by the development of computers (especially rounding problems) and it has lead into the interval arithmetics [7]. Recall, for example, that the summation in this case is given by

$$[a,b]+[c,d] = \{x + y | \, x \in [a,b],\, y \in [c,d]\} = [a+c, b+d].$$

A similar idea has lead Aumann [1] to introduce his integral of set–valued functions. Both these concepts are of the same nature as Zadeh's extension principle [12] is, and in the framework of Choquet integral they appear in several works, see e.g. [6, 13]. We recall here the discrete version of this interval–valued Choquet integral.

Definition 1. Let $F : U \to J([0,1])$ be an interval–valued fuzzy set, and $m : 2^U \to [0,1]$ a fuzzy measure.

Choquet integral–based expectation $\mathbf{C}_m(F)$ is given by

$$\begin{aligned}\mathbf{C}_m(F) &= \{C_m(f) | f : U \to [0,1],\ f(u_i) \in F(u_i)\} = \\ &= [C_m(f_*),\ C_m(f^*)]\,, \end{aligned} \quad (2)$$

where $f_*, f^* : U \to [0,1]$ are given by $f_*(u_i) = a_i$ and $f^*(u_i) = b_i$, with $[a_i,\ b_i] = F(u_i)$.

Several properties of the discrete interval–valued Choquet integral $\mathbf{C}_m$ are discussed and introduced in [6, 13]. For example, this integral is comonotone additive,

$$\mathbf{C}_m(F+G) = \mathbf{C}_m(F) + \mathbf{C}_m(G)$$

whenever $F, G : U \to J([0,1])$ are such that $F(u_i) + G(u_i) \subseteq [0,1]$ for each $u_i \in U$, and F, G are comonotone, i.e., $(f^*(u_i) - f^*(u_j))\ (g^*(u_i) - g^*(u_j)) \geq 0$ and $(f_*(u_i) - f_*(u_j))\ (g_*(u_i) - g_*(u_j)) \geq 0$ for all $u_i, u_j \in U$.

3 ALTERNATIVE DISCRETE INTERVAL–VALUED CHOQUET INTEGRALS

The idea of a discrete Choquet integral C_m, see (1), is based on a permutation $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ forcing $F(u_{\sigma(1)}) \leq F(u_{\sigma(2)}) \leq \cdots \leq F(u_{\sigma(n)})$. This idea can be adapted for the interval case only if there is a linear order $\preceq$ on $J([0,1])$. We introduce now a class of such orders.

Lemma 1. *Let $A, B : [0,1]^2 \to [0,1]$ be two aggregation functions [3] such that $A(x,y) = A(u,v)$ and $B(x,y) = B(u,v)$ can happen only if $(x,y) = (u,v)$. Define a relation $\preceq_{A,B}$ on $J([0,1])$ by*

$$[x,y] \preceq_{A,B} [u,v] \text{ whenever } A(x,y) < A(u,v)$$

or

$$A(x,y) = A(u,v) \text{ and } B(x,y) \leq B(u,v).$$

Then $\preceq_{A,B}$ is a linear order on $J([0,1])$ with the minimal element $\{0\} = [0,0]$, and the maximal element $\{1\} = [1,1]$.

The proof of this lemma is trivial. Note that the linear order $\preceq_{A,B}$ refines the standard partial order $\leq$ on intervals, $[x,y] \leq [u,v]$ whenever $x \leq u$ and $y \leq v$, i.e., $[x,y] \leq [u,v]$ implies $[x,y] \preceq_{A,B} [u,v]$.

Two other linear orders that can be recovered in this way are the lexicographical ones:

$$[x,y] \preceq_{P1} [u,v] \text{ whenever } x < u \text{ or } x = u \text{ and } y \leq v$$

and

$$[x,y] \preceq_{P2} [u,v] \text{ whenever } y < v \text{ or } y = v \text{ and } x \leq u$$

that are obtained by taking $A(x,y) = x$ and $B(x,y) = y$, in the first case $A(x,y) = y$ and $B(x,y) = x$ for the second.

Now, we can introduce discrete interval–valued (A,B)–Choquet integrals.

Definition 2. Let $F : U \to J([0,1])$ be an interval–valued fuzzy set, and $m : 2^U \to [0,1]$ a fuzzy measure. Under the constraints of Lemma 1, the $(A,\ B)$–Choquet integral $\mathbf{C}_m^{A,B}(F)$ is given by

$$\begin{aligned}&\mathbf{C}_m^{A,B}(F) = \\ &= \sum_{i=1}^{n} F(u_{\sigma_{A,B}(i)})(m(\{u_{\sigma_{A,B}(i)},\ldots,u_{\sigma_{A,B}(n)}\}) - \\ &\qquad - m(\{u_{\sigma_{A,B}(i+1)},\ldots,u_{\sigma_{A,B}(n)}\})), \end{aligned} \quad (3)$$

where $\sigma_{A,B} : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is a permutation such that $F(u_{\sigma_{A,B}(1)}) \preceq_{A,B} F(u_{\sigma_{A,B}(2)}) \preceq_{A,B} \cdots \preceq_{A,B} F(u_{\sigma_{A,B}(n)})$.

Observe that if $F(u_i) = [a_i, b_i]$, i = 1, ..., n, then 3 can be rewritten into

$$\begin{aligned}&\mathbf{C}_m^{A,B}(F) = \\ &= \Bigg[\sum_{i=1}^{n} a_{\sigma_{A,B}(i)} \cdot (m(\{u_{\sigma_{A,B}(i)},\ldots,u_{\sigma_{A,B}(n)}\}) - \\ &\qquad - m(\{u_{\sigma_{A,B}(i+1)},\ldots,u_{\sigma_{A,B}(n)}\})), \\ &\quad \sum_{i=1}^{n} b_{\sigma_{A,B}(i)} \cdot (m(\{u_{\sigma_{A,B}(i)},\ldots,u_{\sigma_{A,B}(n)}\}) - \\ &\qquad - m(\{u_{\sigma_{A,B}(i+1)},\ldots,u_{\sigma_{A,B}(n)}\}))\Bigg]\,. \end{aligned} \quad (4)$$

Remark. (i) The concept of an interval–valued (A,B)–Choquet integral $\mathbf{C}_m^{A,B}$ extends the standard discrete Choquet integral. Indeed if $F : U \to J([0,1])$ is singleton–valued, i.e., F is a fuzzy subset of U, then $C_m(F) = \mathbf{C}_m(F) = \mathbf{C}_m^{A,B}(F)$ independently of A, B.

(ii) Observe that for a fixed $F : U \to J([0,1])$ such that f_* and f^* are comonotone, i.e., $(f_*(u_i) - f_*(u_j)) \cdot (f^*(u_i) - f^*(u_j)) \geq 0$ for all $u_i,\ u_j \in U$, for any $A,\ B$ satisfying the constraints of Lemma 1, the discrete Choquet integrals introduced in (2) and (3) coincide, $\mathbf{C}_m(F) = \mathbf{C}_m^{A,B}(F)$. However, in general the integral $\mathbf{C}_m$ cannot be expressed in the form $\mathbf{C}_m^{A,B}$, since representability in terms of the bounds of the considered intervals is not assured.

(iii) The linear order on intuitionistic values based on score function and accuracy function, (see Definition 1 in [8]), can be seen as a linear order $\preceq_{M,G}$ on $J([0,1])$, where M is the arithmetic mean and G is the geometric mean. Note that the above mentioned linear order on intuitionistic values is a background of the discrete Choquet–like integral introduced in [8], though the applied arithmetic operations are different from the summation and multiplication.

4 EXAMPLES AND COMPARISONS

Observe first that for several couples $(A_1, B_1), (A_2, B_2), \ldots,$ the linear orders $\preceq_{A_1,B_1}$, $\preceq_{A_2,B_2}$, $\ldots$, may coincide. Indeed, then $\mathbf{C}_m^{A_1,B_1} = \mathbf{C}_m^{A_2,B_2}$. Consider, for example $\preceq_{Min,Max} \equiv \preceq_{P_1,P_2} \equiv \preceq_{P_1,B}$, where $P_1, P_2 : [0,1]^2 \to [0,1]$ are projections, $P_1(x,y) = x$, $P_2(x,y) = y$ and $B : [0,1]^2 \to [0,1]$ is an arbitrary cancellative aggregation function.

When considering P_1 and P_2, the next interesting relationship can be shown.

Proposition 1. *Let $F : U \to J([0,1])$ and a fuzzy measure $m : 2^U \to [0,1]$ be fixed. Denote $\mathbf{C}_m(F) = [\alpha, \beta]$, $\mathbf{C}_m^{P_1,P_2}(F) = [a,b]$, $\mathbf{C}_m^{P_2,P_1}(F) = [c,d]$. Then $\alpha = a$ and $\beta = d$.*

4.1 AN APPLICATION TO MULTICRITERIA DECISION MAKING

In this section we propose an algorithm that makes use of entropies and the concepts we have presented to determine which is the best alternative between a set of them following some criteria that are provided by one or several experts. Suppose that we are given a set of alternatives $\{A_1, \ldots, A_n\}$ and a set of criteria $\{x_1, \ldots, x_k\}$. Then we can write the following multicriteria decision making (MCDM) matrix:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \\ A_n \end{array} \begin{array}{c} \begin{array}{ccc} \quad x_1 & \quad\ldots\quad & x_k \end{array} \\ \begin{pmatrix} ([\underline{\mu_{A_1}}(x_1), \overline{\mu_{A_1}}(x_1)] & \ldots & [\underline{\mu_{A_1}}(x_k), \overline{\mu_{A_1}}(x_k)] \\ ([\underline{\mu_{A_2}}(x_1), \overline{\mu_{A_2}}(x_1)] & \ldots & [\underline{\mu_{A_2}}(x_k), \overline{\mu_{A_2}}(x_k)] \\ \ldots & \ldots & \\ ([\underline{\mu_{A_n}}(x_1), \overline{\mu_{A_n}}(x_1)] & \ldots & [\underline{\mu_{A_n}}(x_k), \overline{\mu_{A_n}}(x_k)] \end{pmatrix} \end{array}$$

where $[\underline{\mu_{A_i}}(x_j), \overline{\mu_{A_i}}(x_j)]$ denotes the degree to which alternative A_i satisfies criterion x_j. We assume that this satisfaction is expressed in an interval-valued way. Observe that in this way we can understand each alternative as an interval-valued fuzzy set over the referential set of criteria and in such a way that each of the intervals $\mu(A_i)(x_j) = [\underline{\mu_{A_i}}(x_j), \overline{\mu_{A_i}}(x_j)]$ provides the membership value of criteria x_j to the interval valued fuzzy set A_i.

The algorithm that we propose is the following.

1. Fix a linear order over the set $J([0,1])$.

2. Select a fuzzy measure m over the set of criteria $\{x_1, \ldots, x_k\}$.

3. FOR each row $i = 1, \ldots, n$ of the MCDM decision matrix DO

3.1 Order the elements in increasing order $\{x_{(1)}, \ldots, x_{x(k)}\}$ withe respect to their corresponding memberships to the set A_i and using the linear order chosen in Step 1.

3.2 Take $\underline{C(A_i)} = 0$ and $\overline{C(A_i)} = 0$.

3.3 FOR each $j = 1, \ldots k$ DO

3.3.1 $\underline{C(A_i)} = \underline{C(A_i)} + \underline{\mu_{A_i}}(x_{(j)})(m(\{x_{(j)}, \ldots, x_{(k)}) - m(\{x_{(j+1)}, \ldots, x_{(k)})$

3.3.2 $\overline{C(A_i)} = \overline{C(A_i)} + \overline{\mu_{A_i}}(x_{(j)}(m(\{x_{(j)}, \ldots, x_{(k)}) - m(\{x_{(j+1)}, \ldots, x_{(k)})$

ENDFOR

3.4 Take $\mathbf{C}(A_i) = [\underline{C(A_i)}, \overline{C(A_i)}]$.

ENDFOR

4. Choose as best alternative the one for which $\mathbf{C}(A_i)$ is the largest.

Now we present specific examples to show how this algorithm works.

Example 1. This example is taken from [10], who based it in the previous work [5]. In this work, the author considers a set of four alternatives and three criteria with the following MCDM matrix:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & \qquad\qquad x_2 & \qquad\qquad x_3 \end{array} \\ \begin{pmatrix} ([0.45, 0.65] & [0.50, 0.70] & [0.20, 0.45] \\ ([0.65, 0.75] & [0.65, 0.75] & [0.45, 0.85] \\ ([0.45, 0.65] & [0.45, 0.65] & [0.45, 0.80] \\ ([0.75, 0.85] & [0.35, 0.80] & [0.65, 0.85] \end{pmatrix} \end{array}$$

We will take as linear order the following one: $[a,b] \preceq_{MG} [c,d]$ if and only if $a+b < c+d$ or $a+b = c+d$ and $ab \leq cd$. That is, in the notations of the Section 3, we consider the linear order obtained by taking A as the arithmetic mean and B as the geometric mean. This order is the same as the score and accuracy based order.

Now we need to fix the fuzzy measure over the set of criteria that we are going to use. This is a crucial step, and in future works we intend to carry on a deep study on which fuzzy measures are the best fitted ones for a given problem. In this case, we use the following easy fuzzy measure:

$$m(\emptyset) = 0$$
$$m(\{x_1\}) = m(\{x_2\}) = m(\{x_3\}) = \frac{1}{3}$$
$$m(\{x_1, x_2\}) = m(\{x_1, x_3\}) = m(\{x_2, x_3\}) = \frac{2}{3}$$
$$m(\{x_1, x_2, x_3\}) = 1$$

This m can be understood as a measure of how close the measured set is from the total set $\{x_1, x_2, x_3\}$, namely, how far the considered set is from the ideal situation of fulfilling all criteria. Obviously, this is a very course, simplified approach, and it is considered here only to illustrate the way the algorithm works.

If we order with respect to the order $\preceq_{MG}$ the rows of the MCDM matrix, we obtain the following. For A_1:

$$\{x_3, x_1, x_2\}$$

For A_2 we have:

$$\{x_3, x_1, x_2\}$$

For A_3:

$$\{x_1, x_2, x_3\}$$

and finally, for A_4, we arrive at:

$$\{x_2, x_3, x_1\}\ .$$

The corresponding calculations then provide that:

$$\mathbf{C}(A_1) = [0.38, 0.50]$$
$$\mathbf{C}(A_2) = [0.58, 0.78]$$
$$\mathbf{C}(A_3) = [0.45, 0.70]$$
$$\mathbf{C}(A_4) = [0.58, 0.83]$$

So the final ordering of alternatives is A_4, A_2, A_3, A_1. This is not the same ordering obtained in [10], since the first and the second alternatives in that case are interchanged. Nevertheless, this can be explained by the fact that the approach in Ye's work is completely different, since it is based in the use of entropies and correlation, whereas in our case we are only based in aggregation function theory. Of course, here the choice of the measure has been crucial. In this particular, notice that our Choquet integral reduces to the arithmetic mean of the membership intervals under consideration. In fact, the use of symmetric fuzzy measures leads to OWAs in the sense of [2]

Example 2. Let's consider now the following MCDM matrix, taken from [9]:

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} ([0.70, 0.70] & [0.80, 0.90] & [0.90, 0.90] \\ ([0.60, 0.80] & [0.80, 0.80] & [0.80, 0.90] \\ ([0.60, 0.90] & [0.50, 0.90] & [0.80, 0.80] \\ ([0.40, 0.50] & [0.90, 0.90] & [0.40, 0.90] \end{pmatrix} \end{array}$$

Ordering with respect to $\preceq_{MG}$ we obtain, for A_1 $\{x_1, x_2, x_3\}$; for A_2, $\{x_1, x_2, x_3\}$; for A_3, $\{x_2, x_1, x_3\}$; and for A_4, $\{x_1, x_3, x_2\}$. Let's also consider the fuzzy measure proposed in the same paper:

$$m(\emptyset) = 0$$
$$m(\{x_1\}) = m(\{x_2\}) = 0.4\ ;\ m(\{x_3\}) = 0.3$$
$$m(\{x_1, x_2\}) = 0.6\ ;\ m(\{x_1, x_3\}) = m(\{x_2, x_3\}) = 0.8$$
$$m(\{x_1, x_2, x_3\}) = 1\ .$$

So if we carry on the corresponding calculations, we arrive at:

$$\mathbf{C}(A_1) = [0.81, 0.86]$$
$$\mathbf{C}(A_2) = [0.76, 0.83]$$
$$\mathbf{C}(A_3) = [0.64, 0.87]$$
$$\mathbf{C}(A_4) = [0.60, 0.73]$$

So the final ordering of alternatives is A_1, A_2, A_3, A_4, which is the same ordering obtained in Xu's paper. But if now we consider the order $\preceq_{P2}$, we have that the ordering in each alternative is for A_1 $\{x_1, x_2, x_3\}$; for A_2, $\{x_1, x_2, x_3\}$; for A_3, $\{x_3, x_2, x_1\}$; and for A_4, $\{x_1, x_3, x_2\}$. The calculations of the Choquet integrals for each alternative provide:

$$\mathbf{C}(A_1) = [0.81, 0.86]$$
$$\mathbf{C}(A_2) = [0.76, 0.83]$$
$$\mathbf{C}(A_3) = [0.66, 0.86]$$
$$\mathbf{C}(A_4) = [0.60, 0.82]$$

so now we obtain the following order of alternatives A_1, A_3, A_2, A_4. So it is clear that the order that it is chosen determines the final ordering of alternatives.

5 CONCLUDING REMARKS

We have introduced a new concept of a discrete interval–valued $(A,\ B)$–Choquet integral, extending the classical concept of the discrete Choquet integral. For further generalizations, interval–valued fuzzy measures can be considered. We have presented an application to multicriteria decision making. We expect several applications of our concept in multicriteria decision support area, in image processing, etc.

For applications the choice of the appropriate fuzzy measure as well as that of the linear order is a key point. We intend to carry on an in-depth study of what properties a fuzzy measure and a linear order should fulfill in order to be the best fitted ones for a given application.

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